2.

Path Integrals in QFT

Read: QGBH §4

First, remember ordinary quantum mechanics

Simple Harmonic Oscillator

$$\langle x_2 | e^{-iHT} | x_1 \rangle = \sum_{\substack{\text{paths } x(t) \\ \text{from } x_1 \to x_2}} e^{iS[x(t)]/\hbar}$$

"Lorentzian P.I. calculate transition amplitudes"

wavefn in state $|\psi\rangle$

$$\psi(x) \equiv \langle x | \psi \rangle$$

So the amplitude calculate above is also

$= \psi(x_2, T)$ in state $|x_1\rangle$

QFT ($G_N = 0$, but general curved manifolds)

Again, the Lorentzian path integral calculates a transition amplitude,

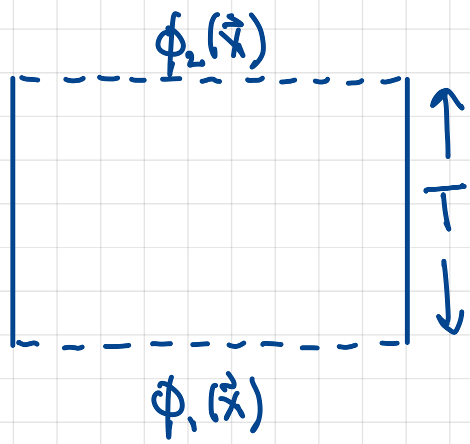
all fields

space only

$$\langle \phi_2(\vec{x}) | \, e^{-iHT} \, | \phi_1(\vec{x}) \rangle$$

$$= \int_{\phi(t=0,\vec{x}) = \phi_1(\vec{x})}^{\phi(t=T,\vec{x}) = \phi_2(\vec{x})} D\phi \; e^{iS[\phi]}$$

$=$



$$= \Psi\left[\phi_2(\vec{x}); T\right] \text{ in state } |\phi_1\rangle$$

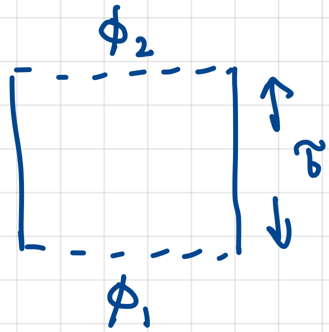
This wavefunctional satisfies the Schrödinger equation, by construction.

Euclidean P.I.

$$\langle \phi_2 | e^{-H\tau} | \phi_1 \rangle = \int_{\phi_1}^{\phi_2} D\phi \; e^{-S_E[\phi]}$$

not unitary

$$=$$

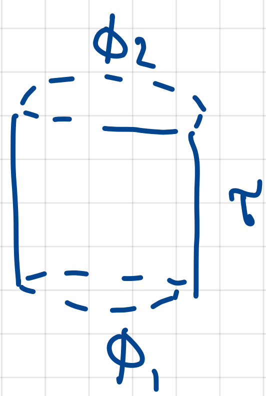


So Euclidean Path integrals also compute transition amplitudes, but in "imaginary time" — really state overlaps.
(not unitary evolution)

Note the special role of the Hamiltonian: it generates evolution from one "time slice" to the next.

The purpose of this lecture is to explain how to think about QFT, quantum states, and thermodynamics with these cartoons.

If space $= S^{D-1}$, then

$$\langle \phi_2 | e^{-\tau H} | \phi_1 \rangle =$$ 

States

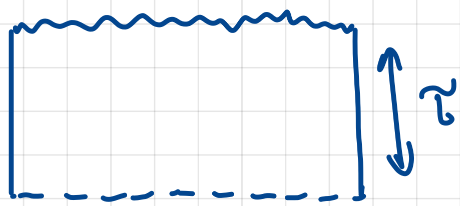
"Cutting" a path integral defines a quantum state:

ex ①

$$|\Psi(\tau)\rangle \equiv e^{-\tau H}|\phi_1\rangle$$

$$= \int_{\phi_1}^{??} D\phi \; e^{-S_E[\phi]}$$

$$=$$



ϕ_1

Formally, a "quantum state" is a linear functional on spatial field configurations, i.e. turns "$\phi(\vec{x})$" $\to$ complex number.

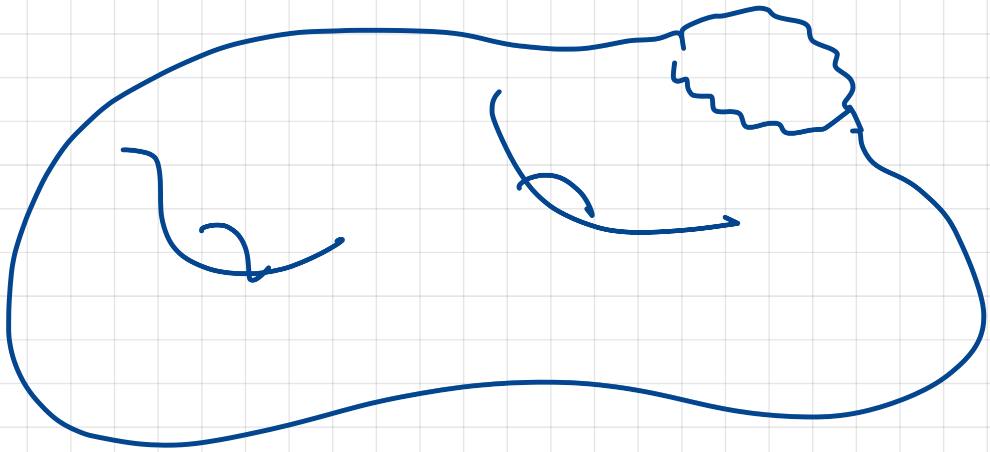
Quantum state ~ linear functional on spatial field data. [caveats...]

$$\Psi[\phi_2] = \langle\phi_2|\Psi\rangle$$

↑

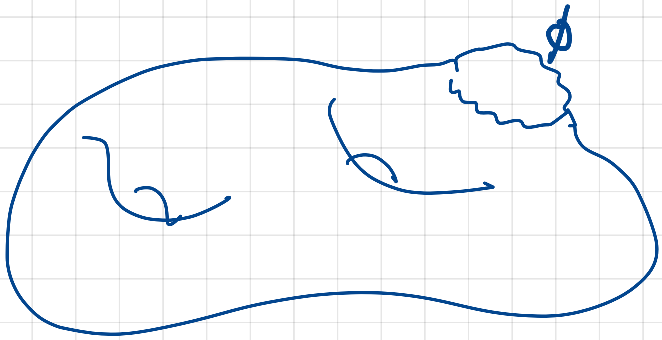
Directs us how to impose boundary condition at the cut.

ex ②

$|C\rangle \equiv$  .

is some crazy state of a 2d QFT on S^1

* Note: $\mathcal{H}$ (Hilbert space) depends only on spatial mf, and does not care about Lorentzian vs. Euclidean

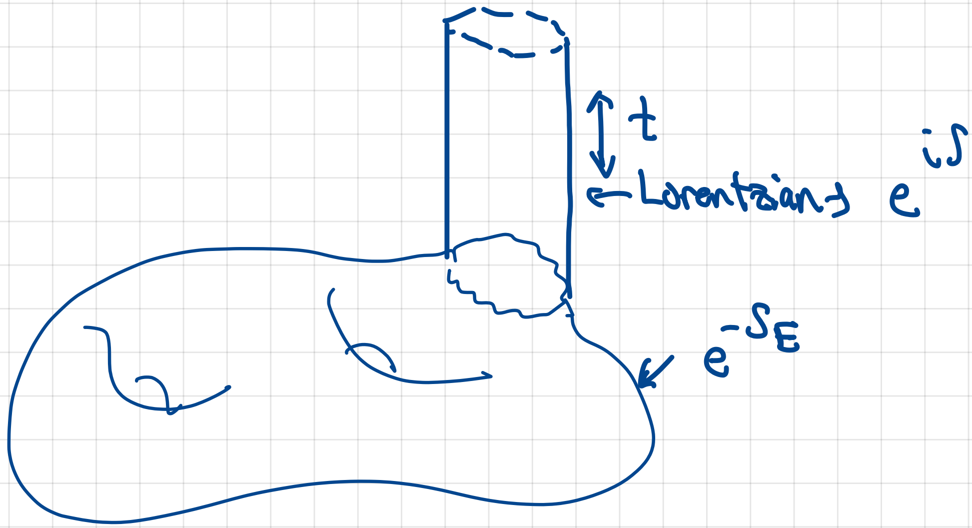
$\Psi_{C.}[\phi] =$ 

$= \int D\phi \, e^{-S_E[\phi]}$ on this mf.

These are Euclidean path integrals, but they define states in the physical Lorentzian theory.

Lorentzian evolution:

$$|C(t)\rangle =$$



This picture denotes a path integral on Euclidean manifold glued to a Lorentzian manifold.

Euclidean P.I. "prepares a state"
Lorentzian P.I. evolves in t.

ex ③ Ground State

Expand any state in energy eigenstates

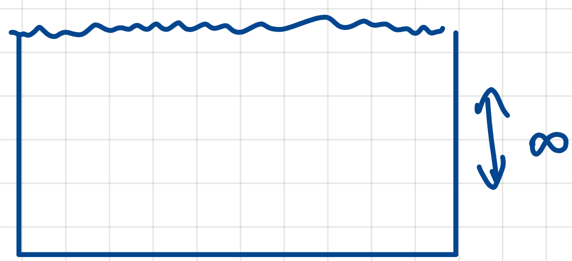
$$|\Psi\rangle = \sum_n \psi_n |n\rangle, \qquad H|n\rangle = E_n |n\rangle$$

Euclidean evolution damps all excitations,

$$e^{-\tau H}|\Psi\rangle \approx e^{-\tau E_0}|0\rangle \quad \text{as} \quad \tau \to \infty$$

Therefore,

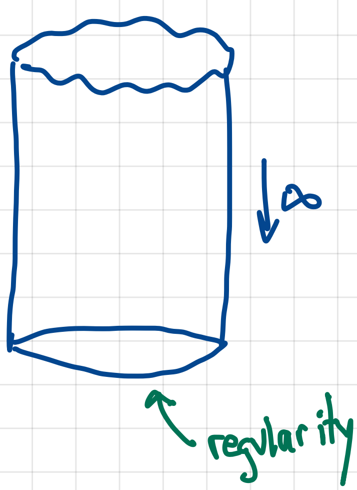
$|0\rangle =$



P.I. on $\mathbb{R}_- \times \mathbb{R}^{D-1}$

(Note: unnormalized)

$|0\rangle_{S'} =$



But what boundary conditions do we put @ $-\infty$?

Answer: <u>regularity</u>.

Any non-singular boundary condition will give $|0\rangle$ because excitations are infinitely damped.

Euclidean Correlators

QFT on space $= \mathbb{R}^{D-1}$

<u>ex.</u>

Euclidean point $(\tau_2, \vec{x}_2)$

$$\langle \phi(x_1)\, \phi(x_2) \rangle$$

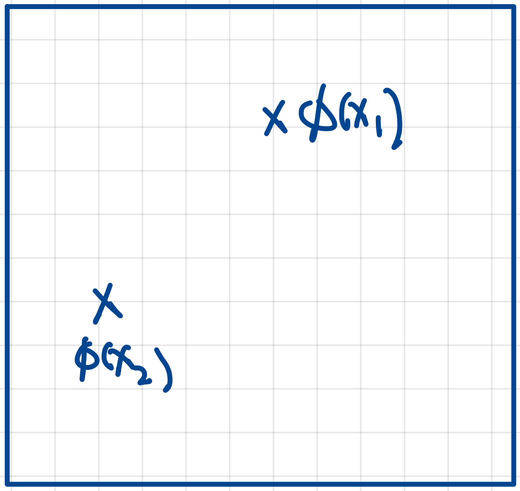
$$= \langle 0 | \phi(x_1) \phi(x_2) | 0 \rangle$$

field operators

$$= \int D\phi \;\; \phi(x_1)\phi(x_2)\, e^{-S_E}$$

numbers

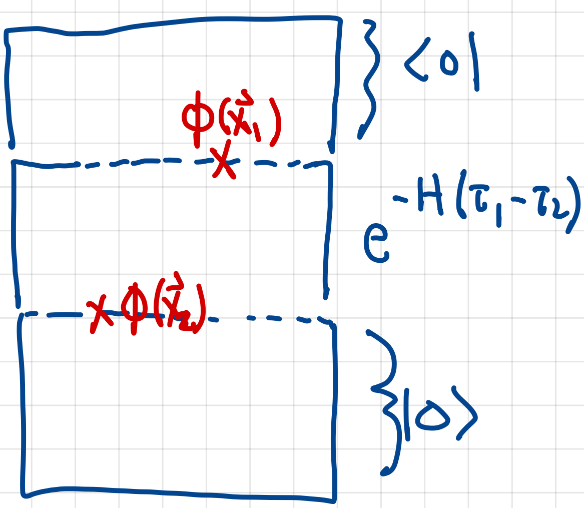
Let's check the path integral agrees with the operator version.

$=$ 

P.I. on full $\mathbb{R}^d$

$$\langle 0|\, \phi(x_1)\phi(x_2)\,|0\rangle$$

$$= \langle 0|\, \cancel{e^{H\tau_1}}\, \phi(\vec{x}_1)\, e^{-H(\tau_1-\tau_2)}\, \phi(\vec{x}_2)\, \cancel{e^{-H\tau_2}}\, |0\rangle$$

$=$ 

I usually read operators Right → Left to make path integrals bottom → top.

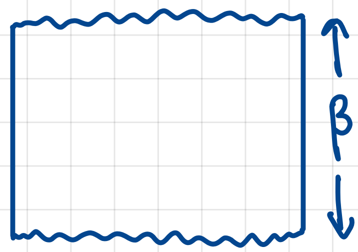
Density Matrix

ex: thermal state

$$\rho = e^{-\beta H}$$

$$\langle \phi_2 | \rho | \phi_1 \rangle \rightsquigarrow 2 \text{ cuts}$$

$\rho =$

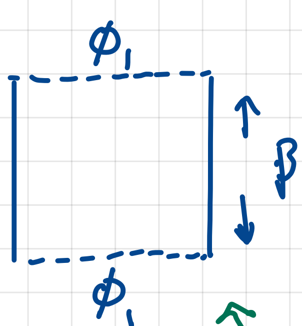


As usual, this picture means "to compute matrix elements, specify boundary conditions at cuts and do the path integral."

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$$

$$= \sum_{\phi_1} \langle \phi_1 | e^{-\beta H} | \phi_1 \rangle$$

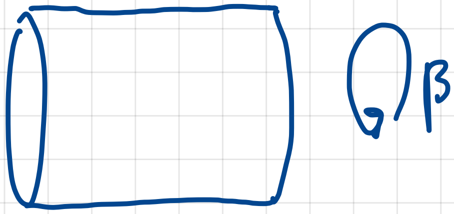
$= \sum_{\phi_1}$



periodic bc!

This is a path integral with periodic boundary conditions, so

$$Z(\beta) =$$



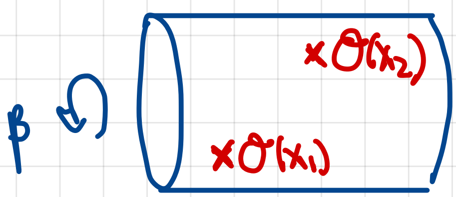
Trace glues cuts together.

Euclidean Thermal Correlator

$$G_\beta = \frac{1}{Z} \operatorname{Tr} e^{-\beta H} \mathcal{O}(x_1)\mathcal{O}(x_2)$$

$$= \langle \mathcal{O}(\tau_1, \vec{x}_1)\, \mathcal{O}(\tau_2, \vec{x}_2) \rangle_\beta$$

$$\propto$$



$$\boxed{G_\beta(t, \vec{x}) = G_\beta(t + i\beta, \vec{x})}$$ "KMS condition"

Finite temp. ~ periodic in imaginary time

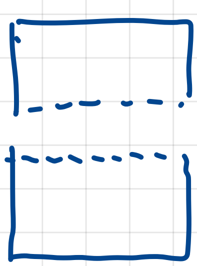
Of course we can also go to Lorentzian. Many of the things a described in vacuum are still true, in particular all real-time correlators are analytic continuations of this Euclidean one.

<u>exercise</u> Derive KMS from trace defn. of G_β

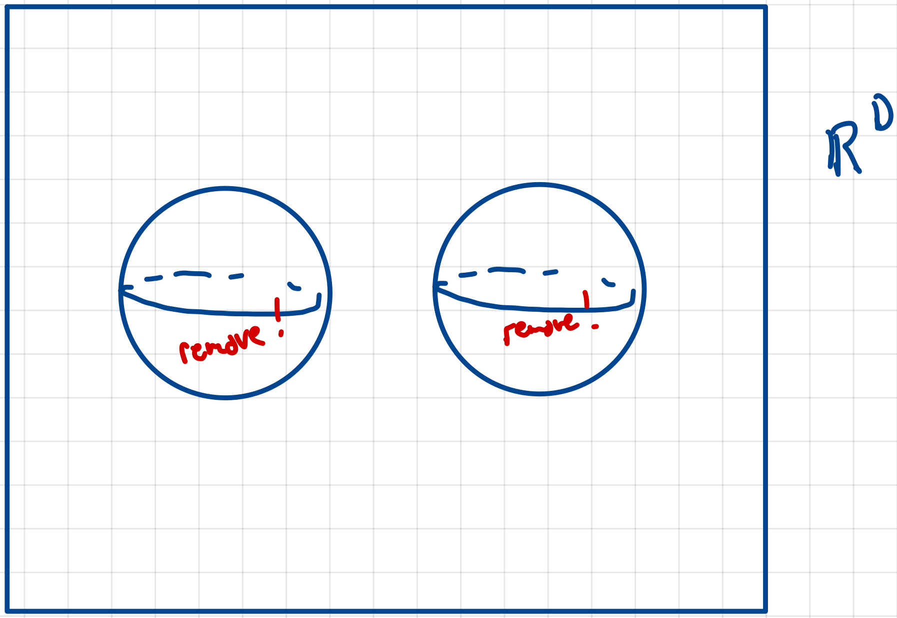
SKIP

Any path integral with 2 identically-shaped cuts defines a density matrix:

<u>ex.</u> $\rho_{vac} = |0\rangle\langle 0|$

$=$ 

ex.



$=$ Some ρ for QFT on S^{D-1}